Cartographic Groups of Regular Toroidal Graphs

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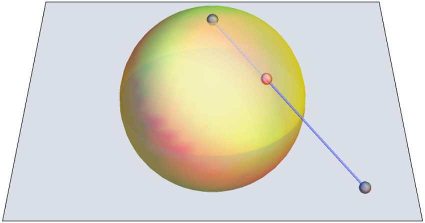
Outline

Riemann Surfaces

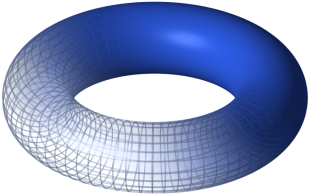
Let X be a compact, connected Riemann surface of genus g. There are two examples of interest.

- The set of complex points, namely $X = S^2(\mathbb{R})$ identified as the Riemann sphere.
- The set of complex points, namely $X = \mathbb{T}^2(\mathbb{R})$, identified as the Torus.

Examples of Riemann Surfaces



Sphere $\mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$



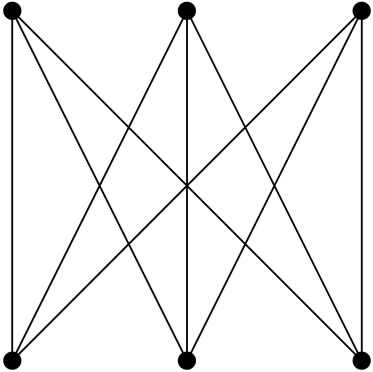
Torus $E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$

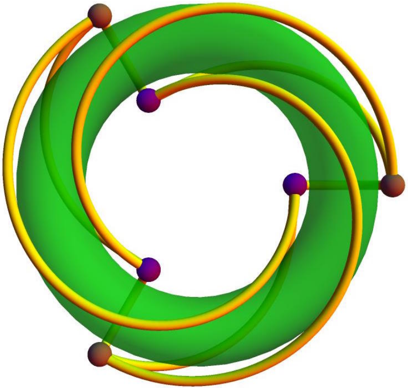
Bipartite Toroidal Graphs

We say that a graph $\Gamma = (V, E)$ is a bipartite, toroidal graph if

- the vertices $V = B \cup W$ are a disjoint union of "black" vertices B and "white" vertices W such that no vertex $P \in B$ ($P \in W$, respectively) is connected by an edge to another vertex $P \in B$ ($P \in W$, respectively), and
- Γ can be embedded in the torus $\mathbb{T}^2(\mathbb{R})$ without any of its edges crossing but cannot be embedded in sphere $S^2(\mathbb{R})$ in a similar fashion.

Example of a Toroidal Graph





Utility Graph

Degree Sequences

Let $\Gamma = (V, E)$ be a bipartite, toroidal graph. We denote the Degree sequence of Γ as the multiset of multisets

$$\mathcal{D} = \left\{ \left\{ e_P \,\middle|\, P \in B \right\},\ \left\{ e_P \,\middle|\, P \in W \right\},\ \left\{ e_P \,\middle|\, P \in F \right\} \right\}$$

where e_P is:

- the number of edges emanating from each "black" vertex when $P \in B$
- the number of edges emanating from each "white" vertex when $P \in B$
- "white" vertices surrounding each face for $P \in F$

The number of edges of Γ is given by the Degree Sum Formula

$$N = \sum_{P \in B} e_P = \sum_{P \in W} e_P = \sum_{P \in F} e_P = |B| + |W| + |F|.$$

Degree Sequences for Toroidal graphs with $N \leq 5$.

Degree N	Degree Sequences $\mathcal{D}$
$N = 1$	None
$N = 2$	None
$N = 3$	$\{\{3\}, \{3\}, \{3\}\}$
$N = 4$	$\{\{1,3\}, \{4\}, \{4\}\}$ $\{\{2,2\}, \{4\}, \{4\}\}$

Degree N	Degree Sequences $\mathcal{D}$
$N = 5$	$\{\{1, 1, 3\}, \{5\}, \{5\}\}$ $\{\{1, 2, 2\}, \{5\}, \{5\}\}$ $\{\{1, 4\}, \{1, 4\}, \{5\}\}$ $\{\{2, 3\}, \{2, 3\}, \{5\}\}$ $\{\{2, 3\}, \{1, 4\}, \{5\}\}$

Cartographic Group of Γ

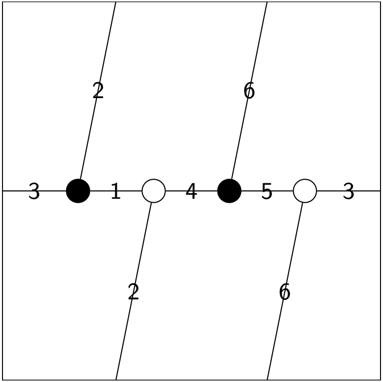
In 2013, Klug, Musty, Schiavone, and Voight gave an algorithm to obtain a triple $(\sigma_0, \sigma_1, \sigma_\infty)$ which are elements of S_N and can be expressed as a disjoint product of cycles of cycle type $\mathcal{D}$ such that:

- $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$
- $G = \left\langle \sigma_0, \sigma_1, \sigma_\infty \right\rangle$ is a transitive subgroup of S_N.

The transitive subgroup G of S_N is what we define as the Cartographic group of the graph Γ.

How to Draw Toroidal Graphs from a Triple

$$\sigma_0 = (1\ 2\ 3)(4\ 5\ 6)$$
$$\sigma_1 = (1\ 2\ 4)(3\ 5\ 6)$$
$$\sigma_\infty = (1\ 6\ 3)(2\ 4\ 5)$$

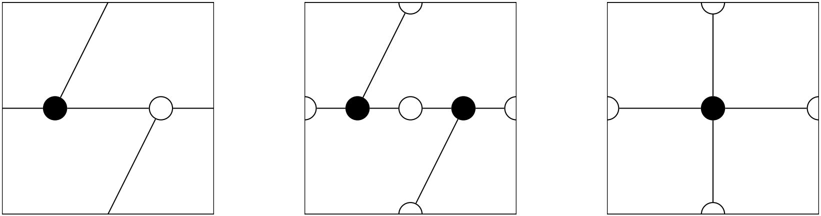


Regular Toroidal Graphs

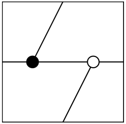
We focus on graphs which cannot be embedded on the sphere $S^2(\mathbb{R})$ without crossings, but which can be embedded on the torus $\mathbb{T}^2(\mathbb{R})$ without crossings.

A regular toroidal graph is a graph where there are the same number of edges incident to every black vertex, and there are the same number of edges incident to every white vertex. Every regular toroidal graph $\Gamma \hookrightarrow \mathbb{T}^2(\mathbb{R})$ must have one of the degree sequences:

$$[3^n;\, 3^n;\, 3^n], \quad [3^{2n};\, 2^{3n};\, 6^n], \quad [4^n;\, 2^{2n};\, 4^n]$$



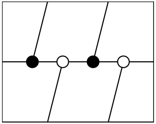
Base Cases for $[3^n; 3^n; 3^n]$



n=1

$\sigma_0 = (1\,2\,3)$

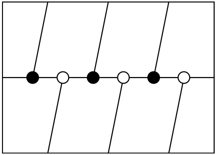
$\sigma_1 = (1\,2\,3)$



n=2

$\sigma_0 = (1\,2\,3)\,(4\,5\,6)$

$\sigma_1 = (1\,2\,4)\,(3\,5\,6)$



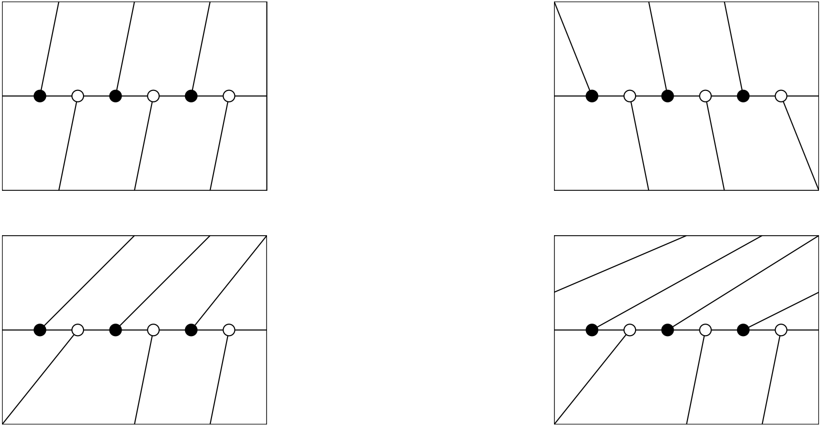
n=3

$\sigma_0 = (1\,2\,3)\,(4\,5\,6)\,(7\,8\,9)$

$\sigma_1 = (1\,2\,4)\,(3\,7\,8)\,(5\,6\,9)$

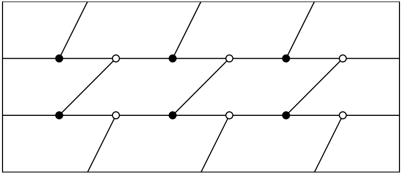
Example: n=3

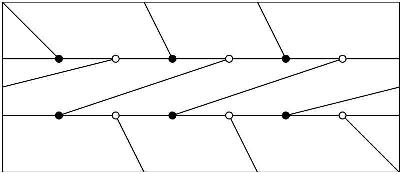
For $[3^3; 3^3; 3^3]$ there are 2 distinct regular toroidal graphs.



Upper Bound on Number of Graphs

$[3^6; 3^6; 3^6]$





Corollary: By rotating a line in a $p \times q$ box you never change the total sum of all the skip values in every region $mod q$. Therefore for the $p \times q$ graphs for any composite n where $p, q > 1$, the sum of the nontrivial, proper divisors of n is an upper bound.

Cartographic Groups of Regular Toroidal Graphs

- Motivating Question: What are the possible cartographic groups of Regular Toroidal Graphs?
- Note: By looking at cycle types, it can be seen that the cartographic groups for the $[3^{2n}; 2^{3n}; 6^n]$, $[3^n; 3^n; 3^n]$, and $[4^n; 2^{2n}; 4^n]$ passports have generators σ_0,σ_1, and σ_∞, satisfying the following respective relations:
- $\sigma_0^3 = \sigma_1^2 = \sigma_\infty^6 = \sigma_0\sigma_1\sigma_\infty = 1$
- $\sigma_0^3 = \sigma_1^3 = \sigma_\infty^3 = \sigma_0\sigma_1\sigma_\infty = 1$
- $\sigma_0^4 = \sigma_1^2 = \sigma_\infty^4 = \sigma_0\sigma_1\sigma_\infty = 1$
- These generators can be swapped out for different generators a, b, and c which have some convenient properties.

Main Results for $[3^{2n}; 2^{3n}; 6^n]$ and $[3^n; 3^n; 3^n]$

For the Cartographic groups with the passport $[3^{2n}; 2^{3n}; 6^n]$ (resp. $[3^n; 3^n; 3^n]$), suppose the order of a and b, m, is divisible only by primes p_1 satisfying $p_1 \equiv -1 \pmod 6$ (resp. $p_1 \equiv -1 \pmod 3$)

Then, we have that $G \simeq \left(Z_m \times Z_m\right) \rtimes Z_6$ (resp. $G \simeq \left(Z_m \times Z_m\right) \rtimes Z_3$)

Question: What is the relationship between the number of faces, n, and the order of a and b, m?

Determining the Cartographic Groups of Regular Toroidal Graphs with the Passport $[3^{2n}, 2^{3n}, 6^n]$

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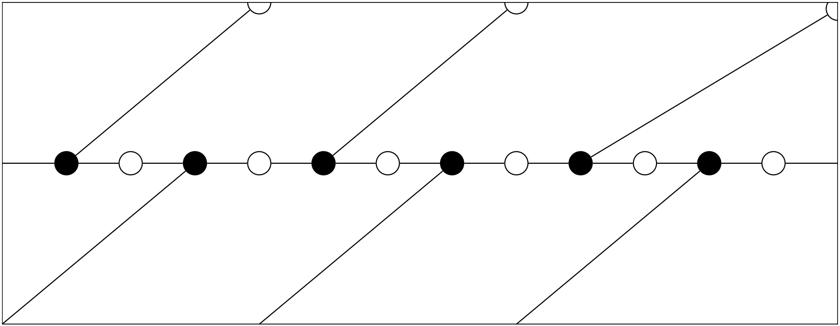

Figure: Horizontal Toroidal Graph with $n = 3$ and $k = 2$

Determining the Cartographic Groups of Regular Toroidal Graphs with the Passport $[3^{2n}, 2^{3n}, 6^n]$

Suppose we have a horizontal toroidal graph of length n and skip value k.

Conjecture: Fix $m \in \mathbb{N}$, and let $k = 2m$. Let $d = gcd(m^2 + m + 1, n)$. Then the cartographic group for horizontal graphs of length n and skip value k is $\left(Z_n \times Z_{\frac{n}{d}}\right) \rtimes Z_6$.

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Example: The previous graph had $n = 3$ and $k = 2$. Hence $m = 1$, i.e. $m^2 + m + 1 = 3$. So $d = 3$ and $\frac{n}{d} = 1$. Thus the cartographic group of the previous graph is $\left(Z_3 \times Z_1\right) \rtimes Z_6$.

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Evidence: We've tested $k = 2$ to $k = 18$ up to $n = 200$, and we've tested $k = 20$ to $k = 32$ up to $n = 300$.

Determining the Cartographic Groups of Regular Toroidal Graphs with the Passport $[3^{2n}, 2^{3n}, 6^n]$

In the specific cases of $k = 0$ and $k = 2$, we have proofs.

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Proposition: The cartographic group for horizontal graphs of skip value 0 is $\left(Z_n \times Z_n\right) \rtimes Z_6$.

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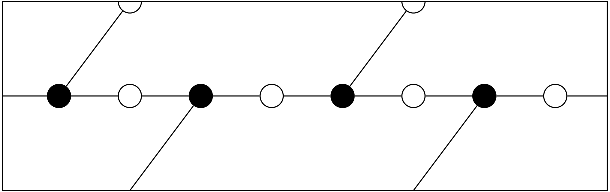

Figure: Toroidal Graph for the passport $[3^4, 2^6, 6^2]$, with $n = 2$ and $k = 0$

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In the specific cases of $k = 0$ and $k = 2$, we have proofs.

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In the specific cases of $k = 0$ and $k = 2$, we have proofs.

Proposition: The cartographic group for horizontal graphs of skip value 2 is $\left(Z_n \times Z_n\right) \rtimes Z_6$ when $3 \nmid n$, and it is $\left(Z_n \times Z_k\right) \rtimes Z_6$ when $n = 3k$.

Determining the Cartographic Groups of Regular Toroidal Graphs with the Passport $[3^{2n}, 2^{3n}, 6^n]$

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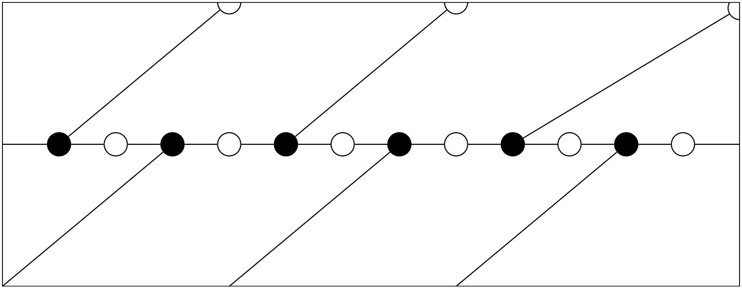

Figure: Toroidal Graph for the passport $[3^6, 2^9, 6^3]$, with $n = 3$ and $k = 2$

Determining the Cartographic Groups of Regular Toroidal Graphs with the Passport $[3^{2n}, 2^{3n}, 6^n]$

Suppose we have a p by q box.

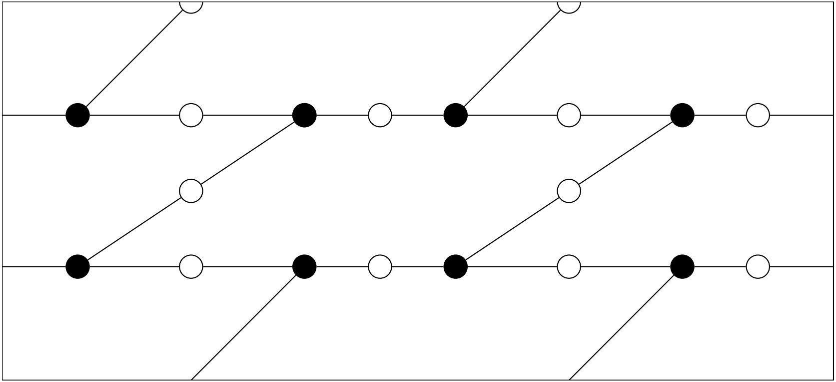

Figure: Toroidal Graph Corresponding to $[3^8, 2^{12}, 6^4]$ with p $=2$ and q $=2$

Determining the Cartographic Groups of Regular Toroidal Graphs with the Passport $[3^{2n}, 2^{3n}, 6^n]$

Suppose we have a p by q toroidal graph.

Conjecture: Let G be the cartographic group of a p x q rectangle of skip value 0. Then $G = H \rtimes K$ where $K \cong Z_6$ and $H \cong Z_m \times Z_m$ with $m = lcm(p, q)$.

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Example:

(p, q)	$p\,q$	$gcd(p, q)$	H
(8, 1)	8	1	$Z_8 \times Z_8$
(8, 2)	16	2	$Z_8 \times Z_8$
(8, 3)	24	1	$Z_{24} \times Z_{24}$
(8, 4)	32	4	$Z_8 \times Z_8$

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Evidence: For all $p \in \{1, \ldots, 9\}$ we computed the group of p by q where we varied q from 1 to p.

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Thank You! Questions?